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LETTER TO THE EDITOR

An $SU(2)$ symmetry of the one-dimensional spin-1 XY model

Atsuhiro Kitazawa[1], **Keigo Hijii**[2] **and Kiyohide Nomura**[2]

[1] Research Institute for Applied Mechanics, Kyushu University, 6-1 Kasuga-koen, Kasuga, Fukuoka 816-8580, Japan
[2] Department of Physics, Kyushu University, Hakozaki, Higashi-ku, Fukuoka 812-8581, Japan

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Abstract
We show that the one-dimensional spin-1 XY model has an additional $SU(2)$ symmetry for the open boundary condition and for an artificial one. We can explain some degeneracies of excitation states which were reported in previous numerical studies.

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In the one-dimensional (1D) spin-1 ($S = 1$) XXZ model, a quantum phase transition between the critical-XY and the Haldane phases takes place at the point of the XY model. This fact has been argued by some authors [1–5]. An important feature is the existence of degeneracies of excitation states with different total magnetization of S^z-direction M. Alcaraz and Moreo [2] studied the XXZ model with $S = 1, 3/2, 2, 5/2$ and 3 numerically, and found the behaviour of critical exponents. From the critical theory and the function of the critical exponent for the $S = 1$ case, it was indicated that the phase boundary between the critical-XY and the Haldane phases is the XY model point. At this point, they observed degeneracies for the open boundary case among the first excited states of $M = 0$ and of $M = \pm 2$, and among the third excited state of $M = \pm 1$ and the first excited state of $M = \pm 3$. With the periodic boundary condition, Kitazawa *et al* [3] studied the quantum phase transition of the 1D $S = 1$ bond-alternating XXZ model numerically, and found that there exists a degeneracy among states with $M = 0$ and $M = \pm 4$ on the line of the XY model. Moreover, Nomura and Kitazawa [5] found that a state with $M = 0$ of the 1D $S = 1$ XY model with the twisted boundary condition has the same energy as states with $M = \pm 2$ of the model with the periodic boundary condition. The authors of [2–5] argued that the above mentioned degeneracies at the XY model point is evidence for the phase boundary. Thus, we think that these degeneracies are due to some symmetry of the 1D $S = 1$ XY model.

In this paper, we show that the 1D $S = 1$ XY model has an additional $SU(2)$ symmetry for the open boundary case and for an artificial boundary case, and explain the above mentioned degeneracies of excitation states.

We consider the following 1D $S = 1$ XY model:

$$H_{XY} = \sum_{j=1}^{L-1} J_{(j,j+1)}\left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y\right) + J_{(L,1)}\left(S_L^x S_1^x + S_L^y S_1^y\right)$$
$$= \sum_{j=1}^{L-1} \frac{J_{(j,j+1)}}{2}\left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right) + \frac{J_{(L,1)}}{2}\left(S_L^+ S_1^- + S_L^- S_1^+\right) \tag{1}$$

where S_j^x, S_j^y and S_j^z $\left(S_j^\pm = S_j^x \pm \mathrm{i}S_j^y\right)$ are spin-1 operators at the jth site, L is the system size, and couplings of the neighbouring sites $J_{(j,j+1)}$ are arbitrarily distributed. Note that this model does not have the usual $SU(2)$ symmetry relating to the total spin operator $S_\mathrm{T}^a = \sum_{j=1}^L S_j^a$ $(a = x, y, z)$.

Firstly, let us define the following operators:

$$\tilde{s}_j^\pm = \tfrac{1}{2}\left(S_j^\pm\right)^2 \qquad \tilde{s}_j^z = \tfrac{1}{2}S_j^z. \tag{2}$$

These operators satisfy the commutation relation

$$\left[\tilde{s}_j^z, \tilde{s}_k^\pm\right] = \pm\delta_{jk}\tilde{s}_j^\pm \tag{3}$$

and using

$$\left[\left(S_j^+\right)^2, \left(S_j^-\right)^2\right] = -8\left(S_j^z\right)^3 + 4(2S^2 + 2S - 1)S^z \tag{4}$$

and $\left(S_j^z\right)^3 = S_j^z$ for $S = 1$, we also have

$$\left[\tilde{s}_j^+, \tilde{s}_k^-\right] = 2\delta_{jk}\tilde{s}_j^z. \tag{5}$$

Thus operators $\tilde{s}_j^\pm$ and $\tilde{s}_j^z$ form a basis of $su(2)$ algebra. At a single site, we have $\frac{1}{2}\left(\tilde{s}_j^+\tilde{s}_j^- + \tilde{s}_j^-\tilde{s}_j^+\right) + \left(\tilde{s}_j^z\right)^2 = \frac{3}{4}\left(S_j^z\right)^2$, and the state with $S_j^z = 0$ corresponds to spin-0 state and the states with $S_j^z = \pm 1$ correspond to spin-1/2 states for the operator (2). The operator $\sum_j \tilde{s}_j^z$ commutes with the Hamiltonian, but operators $\sum_j \tilde{s}_j^\pm$ do not.

Next, we introduce new operators

$$s_j^\pm = \tfrac{1}{2}\left(S_j^\pm\right)^2 U_j \qquad s_j^z = \tfrac{1}{2}S_j^z\left(=\tilde{s}_j^z\right) \tag{6}$$

where U_j is the following non-local unitary operator:

$$U_1 = 1 \qquad \text{and} \qquad U_j = \prod_{l=1}^{j-1}\left(1 - 2\left(S_l^z\right)^2\right) = \mathrm{e}^{\mathrm{i}\pi\sum_{l=1}^{j-1} S_l^z} \qquad \text{for} \quad j > 1. \tag{7}$$

We show that operators (6) obey the commutation relation of $su(2)$, and that the total operators $s_\mathrm{T}^\pm = \sum_{j=1}^L s_j^\pm$ and $s_\mathrm{T}^z = \sum_{j=1}^L s_j^z$ commute with the Hamiltonian (1) with the open boundary condition ($J_{(L,1)} = 0$). From the commutation relation

$$\left[\left(S_j^z\right)^2, \left(S_j^\pm\right)^2\right] = 4\left(S_j^\pm\right)^2\left(1 \pm S_j^z\right) = 0 \tag{8}$$

(in which the second equality is valid for the $S = 1$ case), the factor U_j does not affect the commutativity among the new operators $\left(s_j^\pm, s_j^z\right)$ at different sites. Thus, we obtain the commutation relation of $su(2)$

$$\left[s_j^z, s_k^\pm\right] = \pm\delta_{jk}s_j^\pm \qquad \left[s_j^+, s_k^-\right] = 2\delta_{jk}s_j^z \tag{9}$$

and

$$\left[s_\mathrm{T}^z, s_\mathrm{T}^\pm\right] = \pm s_\mathrm{T}^\pm \qquad \left[s_\mathrm{T}^+, s_\mathrm{T}^-\right] = 2s_\mathrm{T}^z. \tag{10}$$

We also have the relation between the original spin-1 operator $S_j^{\pm}$ and U_k as

$$S_j^{\pm} U_k = \begin{cases} -U_k S_j^{\pm} & j < k \\ U_k S_j^{\pm} & j \geqslant k. \end{cases} \tag{11}$$

The factor (7) appears in the string order parameter of the Haldane-gap state [1] and relates to the Jordan–Wigner transformation. From equation (11), we have a Jordan–Wigner-type transformation from spin-1 operators to spin-1/2 fermions excluding double occupancy

$$\begin{aligned} \tilde{c}_{j,\uparrow}^{\dagger} &= \frac{1}{\sqrt{2}} S_j^z S_j^+ U_j \ \left(= \frac{1}{\sqrt{2}} S_j^+ \left(1 - \left(S_j^z\right)^2\right) U_j \right) \\ \tilde{c}_{j,\downarrow}^{\dagger} &= -\frac{1}{\sqrt{2}} S_j^z S_j^- U_j \ \left(= \frac{1}{\sqrt{2}} S_j^- \left(1 - \left(S_j^z\right)^2\right) U_j \right) \end{aligned} \tag{12}$$

where $\tilde{c}_{j,\sigma}^{\dagger} = c_{j,\sigma}^{\dagger}\left(1 - c_{j,-\sigma}^{\dagger} c_{j,-\sigma}\right)$ with the usual spin-1/2 fermion operators $c_{j,\sigma}^{\dagger}$ and $c_{j,\sigma}$. In the fermion system, the spin operator is given by equation (2),

$$\tilde{c}_{j,\uparrow}^{\dagger} \tilde{c}_{j,\downarrow} = \tilde{s}_j^+ \qquad \tilde{c}_{j,\downarrow}^{\dagger} \tilde{c}_{j,\uparrow} = \tilde{s}_j^- \qquad \tfrac{1}{2}\left(\tilde{c}_{j,\uparrow}^{\dagger} \tilde{c}_{j,\uparrow} - \tilde{c}_{j,\downarrow}^{\dagger} \tilde{c}_{j,\downarrow}\right) = \tilde{s}_j^z \tag{13}$$

and the number operator is $\tilde{c}_{j\uparrow}^{\dagger} \tilde{c}_{j\uparrow} + \tilde{c}_{j\downarrow}^{\dagger} \tilde{c}_{j\downarrow} = \left(S_j^z\right)^2$.

Let us see the commutativity of the operator $s_{\mathrm{T}}^{\pm}$ and the Hamiltonian with the open boundary condition. From the fact that $\left[s_k^{\pm}, S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right] = 0$ for $k \neq j$ and for $k \neq j+1$, it is sufficient to see whether

$$\left[s_j^+, S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right] + \left[s_{j+1}^+, S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right] \tag{14}$$

is zero or not. For the first commutation relation, we have

$$\left[s_j^+, S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right] = \tfrac{1}{2} S_{j+1}^+ \left[\left(S_j^+\right)^2, S_j^-\right] U_j = S_{j+1}^+ S_j^+ \left(1 + 2S_j^z\right) U_j = S_{j+1}^+ S_j^+ U_{j+1} \tag{15}$$

where we used an identity

$$S_j^+ \left(1 + 2S_j^z\right) = S_j^+ \left[1 - 2\left(S_j^z\right)^2\right]$$

for the spin-1 operators. We calculate the second commutation relation of (14) as

$$\begin{aligned} \left[s_{j+1}^+, S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right] &= \left[s_{j+1}^+, S_j^+ S_{j+1}^-\right] \\ &= -\tfrac{1}{2} S_j^+ U_{j+1} \left\{\left(S_{j+1}^+\right)^2 S_{j+1}^- + S_{j+1}^- \left(S_{j+1}^+\right)^2\right\} = -S_{j+1}^+ S_j^+ U_{j+1} \end{aligned} \tag{16}$$

where we used relation (11) and identities for the spin-1 operator $\left(S_j^+\right)^3 = 0$ and

$$\left(S_{j+1}^+\right)^2 S_{j+1}^- + S_{j+1}^- \left(S_{j+1}^+\right)^2 = 2S_{j+1}^+.$$

Hence from equations (15) and (16) we find that (14) is zero, and that the Hamiltonian (1) and the new operators $s_{\mathrm{T}}^{\pm}$ and s_{T}^z commute for the open boundary case. This means that the Hamiltonian is invariant under an $SU(2)$ transformation generated by the operators $s_{\mathrm{T}}^{\pm}$ and s_{T}^z. From this symmetry, there can exist degeneracies between states with $S_{\mathrm{T}}^z = M$ and $M - 2$. When $|E, M\rangle$ is an eigenstate of the Hamiltonian with an eigenvalue E and of S_{T}^z with an eigenvalue M,

$$|E, M - 2\rangle = \frac{1}{\mathcal{N}} s_{\mathrm{T}}^- |E, M\rangle$$

where $\mathcal{N} = \sqrt{(s + M/2)(s - M/2 + 1)}$ is a normalization factor with the length of the 'spin' s for the operators $s_{\mathrm{T}}^{\pm}$ and s_{T}^z is a degenerate eigenstate of the Hamiltonian with an eigenvalue $M - 2$ of S_{T}^z (if $s_{\mathrm{T}}^- |E, M\rangle$ exists). This type of degeneracy was observed in [2].

When interactions $J_{(j,j+1)}$ satisfy the condition $J_{(j,j+1)} = J_{(L-j,L-j+1)}$, the system is invariant under the space inversion $PS_j^{x,y,z}P^{-1} = S_{L-j+1}^{x,y,z}$. From

$$\begin{aligned} Ps_j^{\pm}P^{-1} &= \frac{1}{2}\left(S_{L-j+1}^{\pm}\right)^2 \prod_{l=L-j+2}^{L}\left(1-2\left(S_j^z\right)^2\right) \\ &= \frac{1}{2}\left(S_{L-j+1}^{\pm}\right)^2\left(1-2\left(S_{L-j+1}^z\right)^2\right)U_{L-j+1}\prod_{l=1}^{L}\left(1-2\left(S_l^z\right)^2\right) \\ &= -\frac{1}{2}\left(S_{L-j+1}^{\pm}\right)^2 U_{L-j+1}\,\mathrm{e}^{\mathrm{i}\pi S_\mathrm{T}^z} = -s_{L-j+1}^{\pm}\,\mathrm{e}^{\mathrm{i}\pi S_\mathrm{T}^z} \end{aligned} \tag{17}$$

we obtain

$$P|E, M-2\rangle = -\mathrm{e}^{\mathrm{i}\pi M}\frac{1}{\mathcal{N}}s_\mathrm{T}^- P|E, M\rangle.$$

Hence when M is odd, the degenerate states $|E, M\rangle$ and $|E, M-2\rangle$ have the same eigenvalue of P $(=\pm)$, but when M is even, the two states have different eigenvalues.

For the periodic boundary case, the boundary term

$$\frac{J_{(L,1)}}{2}\left[S_L^+S_1^- + S_L^-S_1^+\right]$$

does not commute with $s_\mathrm{T}^{\pm}$, and the model does not have an $SU(2)$ symmetry. In order to find an $SU(2)$ symmetric boundary term, let us consider the following one:

$$H_{\mathrm{boundary}} = \tfrac{1}{2}J_{(L,1)}\left(S_L^+S_1^-\,\mathrm{e}^{-\mathrm{i}\theta S_\mathrm{T}^z} + S_L^-S_1^+\,\mathrm{e}^{\mathrm{i}\theta S_\mathrm{T}^z}\right) \tag{18}$$

where θ is a real number. The commutation relations between $S_L^+S_1^-\,\mathrm{e}^{-\mathrm{i}\theta S_\mathrm{T}^z} + S_L^-S_1^+\,\mathrm{e}^{\mathrm{i}\theta S_\mathrm{T}^z}$ and s_j^+ are given by

$$\begin{aligned} &\left[s_1^+, S_L^+S_1^-\,\mathrm{e}^{-\mathrm{i}\theta S_\mathrm{T}^z} + S_L^-S_1^+\,\mathrm{e}^{\mathrm{i}\theta S_\mathrm{T}^z}\right] = \tfrac{1}{2}S_L^+\left\{\left(S_1^+\right)^2S_1^- - \mathrm{e}^{-2\mathrm{i}\theta}S_1^-\left(S_1^+\right)^2\right\}\mathrm{e}^{-\mathrm{i}\theta S_\mathrm{T}^z}\Big] \\ &\qquad\times\left[s_j^+, S_L^+S_1^-\,\mathrm{e}^{-\mathrm{i}\theta S_\mathrm{T}^z} + S_L^-S_1^+\,\mathrm{e}^{\mathrm{i}\theta S_\mathrm{T}^z}\right] \\ &\quad= -(1+\mathrm{e}^{-2\mathrm{i}\theta})S_L^+S_1^-s_j^+\,\mathrm{e}^{-\mathrm{i}\theta S_\mathrm{T}^z} - (1+\mathrm{e}^{2\mathrm{i}\theta})S_L^-S_1^+s_j^+\,\mathrm{e}^{\mathrm{i}\theta S_\mathrm{T}^z} \qquad \text{for} \quad 1 < j < L \end{aligned}$$

and

$$\left[s_L^+, S_L^+S_1^-\,\mathrm{e}^{-\mathrm{i}\theta S_\mathrm{T}^z} + S_L^-S_1^+\,\mathrm{e}^{\mathrm{i}\theta S_\mathrm{T}^z}\right] = -\tfrac{1}{2}\left\{\left(S_L^+\right)^2S_L^- + \mathrm{e}^{2\mathrm{i}\theta}S_L^-\left(S_L^+\right)^2\right\}S_1^+\,\mathrm{e}^{\mathrm{i}\theta S_\mathrm{T}^z}U_L$$

where we used $\mathrm{e}^{\pm\mathrm{i}\theta S_j^z}\left(S_j^+\right)^2 = \mathrm{e}^{\pm2\mathrm{i}\theta}\left(S_j^+\right)^2\mathrm{e}^{\pm\mathrm{i}\theta S_j^z}$. If we choose $\theta = \pm\pi/2$, we have

$$\left[s_1^+, S_L^+S_1^-\,\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z} + S_L^-S_1^+\,\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z}\right] = \tfrac{1}{2}S_L^+\left\{\left(S_1^+\right)^2S_1^- + S_1^-\left(S_1^+\right)^2\right\}\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z} = S_L^+S_1^+\,\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z}$$

$$\left[s_j^+, S_L^+S_1^-\,\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z} + S_L^-S_1^+\,\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z}\right] = 0 \qquad \text{for} \quad 1 < j < L$$

and

$$\begin{aligned} \left[s_L^+, S_L^+S_1^-\,\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z} + S_L^-S_1^+\,\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z}\right] &= -\tfrac{1}{2}\left\{\left(S_L^+\right)^2S_L^- - S_L^-\left(S_L^+\right)^2\right\}S_1^+\,\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z}U_L \\ &= -S_L^+S_1^+\,\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z}\prod_{l=1}^{L}\left\{1-2\left(S_l^z\right)^2\right\} \\ &= -S_L^+S_1^+\,\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z}. \end{aligned}$$

Thus, the boundary term (18) commutes with $s_\mathrm{T}^{\pm}$ (and s_T^z) for $\theta = \pm\pi/2$, and accordingly the Hamiltonian of the form

$$H = \sum_{j=1}^{L-1}\frac{J_{(j,j+1)}}{2}\left[S_j^+S_{j+1}^- + S_j^-S_{j+1}^+\right] + \frac{J_{(L,1)}}{2}\left[S_L^+S_1^-\,\mathrm{e}^{\mp\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z} + S_L^-S_1^+\,\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}S_\mathrm{T}^z}\right] \tag{19}$$

commutes with $s_\mathrm{T}^{\pm}$ and s_T^z, and has an $SU(2)$ symmetry.

From the Hamiltonian with the artificial boundary term (19), we can explain the existence of degenerate states for the periodic and the twisted boundary conditions in [3–5]. Let us define

$$H^{(\pm)} = \sum_{j=1}^{L-1} \frac{J_{(j,j+1)}}{2}\left[S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right] \pm \frac{J_{(L,1)}}{2}\left[S_L^+ S_1^- + S_L^- S_1^+\right]$$

and

$$H_1^{(\pm)} = \sum_{j=1}^{L-1} \frac{J_{(j,j+1)}}{2}\left[S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right] \pm \frac{J_{(L,1)}}{2}\left[S_L^+ S_1^- \,\mathrm{e}^{\mp \mathrm{i}\frac{\pi}{2} S_\mathrm{T}^z} + S_L^- S_1^+ \,\mathrm{e}^{\pm \mathrm{i}\frac{\pi}{2} S_\mathrm{T}^z}\right].$$

We regard $H^{(+)}$ as the original Hamiltonian. If $|E, 4N\rangle$ is an eigenstate of the Hamiltonian $H^{(+)}$ ($H^{(-)}$) with an eigenvalue E and of S_T^z with an eigenvalue $4N$ (N is an integer), it is also an eigenstate of $H_1^{(+)}$ $\left(H_1^{(-)}\right)$ because $\mathrm{e}^{\pm \mathrm{i}\frac{\pi}{2} S_\mathrm{T}^z}|E, 4N\rangle = |E, 4N\rangle$. Then if the state

$$|E, 4N-2\rangle = \frac{1}{\mathcal{N}} s_\mathrm{T}^- |E, 4N\rangle$$

(where $\mathcal{N} = \sqrt{(s+2N)(s-2N+1)}$ is a normalization factor with a positive integer s) exists, it is a degenerate eigenstate of the Hamiltonian $H_1^{(+)}$ $\left(H_1^{(-)}\right)$. Since the eigenvalue of S_T^z is $4N-2$ and $\mathrm{e}^{\pm \mathrm{i}\frac{\pi}{2} S_\mathrm{T}^z}|E, 4N-2\rangle = -|E, 4N-2\rangle$, this state is not an eigenstate of $H^{(+)}$ ($H^{(-)}$) but of $H^{(-)}$ ($H^{(+)}$) with an eigenvalue E.

Similarly, if $|E, 4N+2\rangle$ is an eigenstate of $H^{(+)}$ ($H^{(-)}$) with an eigenvalue E and of S_T^z with an eigenvalue $4N+2$, it is an eigenstate of $H_1^{(-)}$ $\left(H_1^{(+)}\right)$. Then if the state

$$|E, 4N\rangle = \frac{1}{\mathcal{N}} s_\mathrm{T}^- |E, 4N+2\rangle$$

(where $\mathcal{N} = \sqrt{(s+2N+1)(s-2N)}$ with a positive integer s) exists, it is a degenerate eigenstate of the Hamiltonian $H_1^{(-)}$ $\left(H_1^{(+)}\right)$. Since the eigenvalue of S_T^z is $4N$, this state is not an eigenstate of $H^{(+)}$ ($H^{(-)}$) but of $H^{(-)}$ ($H^{(+)}$) with an eigenvalue E. Thus, an eigenstate $|E, 2N\rangle$ of the model with the periodic boundary condition ($H^{(+)}$) can have the same energy as the state $|E, 2N \pm 2\rangle$ of the model with the twisted boundary condition ($H^{(-)}$). Let us consider the case that the system is invariant under the space inversion P. From equation (17), we can say that the above states $|E, 2N\rangle$ (an eigenstate of $H^{(+)}$) and $|E, 2N \pm 2\rangle$ (of $H^{(-)}$) have different eigenvalues of P (e.g. if $|E, 2N\rangle$ has $P = +$, then $|E, 2N \pm 2\rangle$ have $P = -$). Examples of this degeneracy were reported in [5].

Although the model with the periodic boundary condition ($H^{(+)}$) does not have the $SU(2)$ symmetry, there exist degenerate states $|E, 2N\rangle$ and

$$|E, 2N-4\rangle = \frac{1}{\mathcal{N}} (s_\mathrm{T}^-)^2 |E, 2N\rangle$$

with even eigenvalues of S_T^z (and with integer eigenvalues of s_T^z). When the system is invariant under the space inversion P, from equation (17) the two states $|E, 2N\rangle$ and $|E, 2N-4\rangle$ have the same eigenvalue of P. When the couplings are uniform $J_{(1,2)} = J_{(2,3)} = \cdots = J_{(L,1)}$, the system is invariant under the translation $T S_j^{x,y,z} T^{-1} = S_{j+1}^{x,y,z}$. In this case, from

$$T\left(s_\mathrm{T}^\pm\right)^2 T^{-1} = \left(s_\mathrm{T}^\pm\right)^2 + 2\sum_{j=2}^{L} s_1^\pm s_j^\pm \left(\mathrm{e}^{\mathrm{i}\pi S_\mathrm{T}^z} - 1\right) \tag{20}$$

the two degenerate states $|E, 2N\rangle$ and $|E, 2N-4\rangle$ have the same eigenvalue of T (i.e. wave number). This type of degeneracy was found in [3].

From equation (8), the following term also commutes with $s_{\mathrm{T}}^{\pm}$ and s_{T}^{z}:

$$H_{\mathrm{Sia}} = \sum_{j=1}^{L} D_j^z \left(S_j^z\right)^2. \tag{21}$$

Hence the Hamiltonians of the form $H_{XY} + H_{\mathrm{Sia}}$ with the open boundary condition and with an artificial boundary condition also have an $SU(2)$ symmetry, and the above argument can also be applied. The phase diagram of the 1D $S = 1$ XXZ model with single ion anisotropy

$$H = \sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z\right) + D \sum_j \left(S_j^z\right)^2 \tag{22}$$

has been studied [1, 6–10]. The phase boundaries between the $XY1$ and the Haldane phases and between the $XY2$ and the Néel phases [8, 9] are on the line $\Delta = 0$ [1, 10], and this reflects the symmetry.

Considering that the 1D $S = 1$ model relates to the $S = 1/2$ two-leg ladder model with strong ferromagnetic inter-chain interactions, we can apply the same analysis for the $S = 1/2$ model [11]

$$\begin{aligned}
H = \sum_{j=1}^{L-1} & J_{(j,j+1)}\left(S_{1,j}^x S_{1,j+1}^x + S_{1,j}^y S_{1,j+1}^y + S_{2,j}^x S_{2,j+1}^x + S_{2,j}^y S_{2,j+1}^y\right) \\
&+ \sum_{j=1}^{L-1} J_{[j,j+1]}\left(S_{1,j}^x S_{2,j+1}^x + S_{1,j}^y S_{2,j+1}^y + S_{2,j}^x S_{1,j+1}^x + S_{2,j}^y S_{1,j+1}^y\right) \\
&+ \sum_{j=1}^{L} J_{xy,j}\left\{S_{1,j}^x S_{2,j}^x + S_{1,j}^y S_{2,j}^y\right\} + \sum_{j=1}^{L} J_{z,j} S_{1,j}^z S_{2,j}^z \\
&+ \frac{J_{(L,1)}}{2}\left(S_{1,L}^+ S_{1,1}^- \,\mathrm{e}^{\mp \mathrm{i}\frac{\pi}{2} S_{\mathrm{T}}^z} + S_{1,L}^- S_{1,1}^+ \,\mathrm{e}^{\pm \mathrm{i}\frac{\pi}{2} S_{\mathrm{T}}^z}\right) \\
&+ \frac{J_{(L,1)}}{2}\left(S_{2,L}^+ S_{2,1}^- \,\mathrm{e}^{\mp \mathrm{i}\frac{\pi}{2} S_{\mathrm{T}}^z} + S_{2,L}^- S_{2,1}^+ \,\mathrm{e}^{\pm \mathrm{i}\frac{\pi}{2} S_{\mathrm{T}}^z}\right) \\
&+ \frac{J_{[L,1]}}{2}\left(S_{1,L}^+ S_{2,1}^- \,\mathrm{e}^{\mp \mathrm{i}\frac{\pi}{2} S_{\mathrm{T}}^z} + S_{1,L}^- S_{2,1}^+ \,\mathrm{e}^{\pm \mathrm{i}\frac{\pi}{2} S_{\mathrm{T}}^z}\right) \\
&+ \frac{J_{[L,1]}}{2}\left(S_{2,L}^+ S_{1,1}^- \,\mathrm{e}^{\mp \mathrm{i}\frac{\pi}{2} S_{\mathrm{T}}^z} + S_{2,L}^- S_{1,1}^+ \,\mathrm{e}^{\pm \mathrm{i}\frac{\pi}{2} S_{\mathrm{T}}^z}\right)
\end{aligned} \tag{23}$$

where $S_{n,j}^a$ $(a = x, y, z)$ is the spin-1/2 operator at the jth site of the leg $n = 1, 2$, and we define $S_{\mathrm{T}}^z = \sum_{j=1}^{L}\left(S_{1,j}^z + S_{2,j}^z\right)$. The intra- and the inter-chain couplings $J_{(j,j+1)}$, $J_{[j,j+1]}$, $J_{xy,j}$ and $J_{z,j}$ are arbitrary, but the model needs the invariance under the exchange of the leg $S_{1,j}^a \leftrightarrow S_{2,j}^a$. In this case, we define the operators

$$s_j^{\pm} = S_{1,j}^{\pm} S_{2,j}^{\pm} \prod_{l=1}^{j-1}\left(-4 S_{1,l}^z S_{2,l}^z\right) \qquad s_j^z = \frac{1}{2}\left(S_{1,j}^z + S_{2,j}^z\right) \tag{24}$$

which satisfy equation (9). The Hamiltonian (23) is invariant under an $SU(2)$ transformation generated by the total operators $s_{\mathrm{T}}^{\pm} = \sum_{j=1}^{L} s_j^{\pm}$ and $s_{\mathrm{T}}^z = \sum_{j=1}^{L} s_j^z$. With the Jordan–Wigner transformation, we can see that the operators (24) relate to the pseudospin operators of the Hubbard model [12–14].

In summary, introducing new operators (6), we considered an additional $SU(2)$ symmetry of the one-dimensional spin-1 XY model. Interactions of the model were assumed for nearest neighbour spins. But the strength of interactions can be arbitrarily distributed, so that the argument can be applied for the random coupling case. The existence of the $SU(2)$ symmetry depends on the boundary term. The symmetry exists for the open boundary case, but does not exist for the periodic boundary case. We found an $SU(2)$ symmetric boundary term depending on the operator S_{T}^{z}. Considering the $SU(2)$ symmetric cases, we explained degeneracies of excitation states which were reported in previous numerical studies.

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